Compton/Thomson backscattering:

Replace undulator magnetic field field with laser e-m field.

Differences to undulator/wiggler:

• The laser field envelope is not a top-hat function, but a function of time → K-parameter is not constant, has to be replaced by dimensionless laser amplitude a(t)

• The field is not stationary, but propagates in space: \( \lambda_u \) depends on interaction angle as \( \lambda_u = \lambda_L / (1 - \beta \cos \theta) \), where \( \theta \) is the collision angle (head-on collision: \( \theta = \pi \), \( \lambda_u = \lambda_L / 2 \))

• The laser wavelength can vary in time \( \lambda_L = f(t) \)

\[
\lambda_{x-ray} = \frac{\lambda_L(t)}{2(1 - \beta \cos \theta) \gamma^2} \left( 1 + \frac{a_0^2(t)}{2} + \gamma^2 \theta^2 \right)
\]
Trajectory of electron in plane monochromatic light wave (e.g. Gibbon, 2005), $a_0 < 1$, $\gamma \gg 1$:

$$x_T(z) = \frac{a_0}{2\gamma k_L} \sin(2k_L z); \quad k_L = \frac{2\pi}{\lambda_L} \quad \left(\text{Undulator: } x_u(z) = \frac{K}{\gamma k_u} \sin(k_u z)\right)$$

$$\Rightarrow \frac{d^2W}{d\omega d\Omega} = \hbar \alpha f \gamma_{z,0}^2 N_L^2 \sum_{n=1}^{\infty} \hat{\omega}^2 R_n \left[ \gamma_{z,0}^2 C_x^2 + C_z^2 \hat{\theta}^2 - 2\gamma_{z,0} C_x C_z \hat{\theta} \cos(\phi) \right]$$

$$\gamma_{z,0} C_x = a_0 \sum_{m=-\infty}^{\infty} J_m(\alpha_z) \left[ J_{n+2m-1}(\alpha_x) + J_{n+2m+1}(\alpha_x) \right], \quad C_z = 2 \sum_{m=-\infty}^{\infty} J_m(\alpha_z) J_{n+2m}(\alpha_x)$$

$$\alpha_z = \frac{n(\hat{\omega} / \hat{\omega}_n) a_0^2 / 4}{1 + a_0^2 / 2 + \hat{\theta}^2}, \quad \alpha_z = \frac{n(\hat{\omega} / \hat{\omega}_n) 2a_0 \cos(\phi)}{1 + a_0^2 / 2 + \hat{\theta}^2}, \quad R_n = \frac{\sin^2 \left[ \pi n N_L (\hat{\omega} / \hat{\omega}_n - 1) \right]}{\left[ \pi n N_L (\hat{\omega} / \hat{\omega}_n - 1) \right]^2}$$

The x-ray photon number per electron in the case for quasi-linear Thomson scattering ($a_0 < 1$) can be derived from integrating the general radiation formula over time and dividing by the photon energy:

$$N_x = \frac{\pi}{3} \alpha_f a_0^2 N_L \approx 10^{-2} a_0^2 N_L \text{ per } e^-$$

Note: $N_L$ is the number of laser oscillation periods, not photons!
For a monochromatic plane wave with 10 oscillations, the Thomson spectra resemble an undulator spectrum (line width $1/NL$, harmonic emission, downshift for $a_0$ approaching unity)

In the case of a Gaussian laser pulse, each oscillation period has a different $a_0$, which progressively downshifts for growing intensity (and shifts back up for falling intensity)
And now the whole enchilada: bunch with energy spread and divergence...
Optimizing Thomson scattering:

\[ N_x = \frac{\pi}{3} \alpha_f a_0^2 N_L \approx 10^{-2} a_0^2 N_L \text{ per } e^- \]

The \( a_0^2 N_L \propto I \tau_L \) - dependence expresses the proportionality to the number of scattering laser photons, so one is free to choose the optimum pulse duration for a given laser energy.

Considerations:
- Spectral brilliance is maximised for \( a_0 \approx 1 \) \( \Rightarrow \) keep \( a_0 \) limited to 1
- best possible overlap between electron bunch and laser pulse \( \Rightarrow \) restricts increasing the laser spot
- Smallest possible electron/laser spot for maximum brilliance \( \Rightarrow \) optimization mainly via pulse stretching/chirping.

On-axis we can always find a pair of \( a(t) \) and \( \lambda(t) \) to fulfil the resonance condition:

\[ a(t) = \sqrt{\frac{8\gamma^2 \lambda_{x-ray}}{\lambda_L(t)} - 2} \]

- requires spectral shaping to match chirp introduced by stretching.
- keep pulse duration shorter than Rayleigh length!
Angular distribution

Deflection angle [mrad]

rad. Int. (a.u.)

Photon Energy (keV)
Experiment on Thomson angle distribution:

Jochmann et al., PRL 111, 114803 (2013)

conventional Linac + high-power scattering laser
All-optical Thomson:  
Experiment at LOA: single beam + plasma mirror (K. TaPhuoc et al, Nature Photonics 6, 308 (2012))

+ Simple, no alignment issues
- weak control over scattering pulse parameters

Experiment at U Nebraska, Lincoln: dual beam, $a_0=0.3$ (N. Powers et al, Nat. Photonics 8, 28 (2014))

+ Complete control
- complicated alignment, high photon energy prohibits high-resolution spectroscopy
Experiment at MPQ:

Shock-front injected e-beams: Electron energy (red – averaged)

X-ray energy (red – averaged; white – expected)

\[ \lambda_{\text{x-ray}} = \frac{\lambda}{4\gamma^2} \left( 1 + \frac{a_0^2}{2} + \gamma^2 \theta^2 \right) \]

Khrennikov et al., PRL 114, 195003 (2015)
Betatron emission: Wiggling in plasma fields:

Longitudinal electric Field $E_z$:

Transverse electric Field $E_r$:

Complete blowout causes homogeneously charged ion background
radial field increases linearly with radius inside blowout zone:

$$E_r = -\frac{\partial_r \phi}{2\varepsilon_0} = \frac{m_e \omega_p^2}{2e} r$$
For an electron with mass $\gamma m_e$, the equation of motion is then

$$E_r = -\partial_r \phi = \frac{en_e}{2\varepsilon_0} r = \frac{m_e \omega_p^2}{2e} r$$

which is a harmonic oscillator with eigenfrequency

$$\omega_\beta = \frac{\omega_p}{\sqrt{2\gamma}} \rightarrow \lambda_\beta \quad \text{some hundreds of microns}$$

$$\Rightarrow \lambda_{x-ray} \approx \frac{\lambda_\beta}{2\gamma^2} \left(1 + \frac{K_\beta^2}{2} + \gamma^2 \Theta^2\right)$$

Note: $\lambda_\beta$ is not constant over time, but depends on $\gamma$ and $n_e$!
First, assume constant gamma:

This oscillation with $\omega_\beta$ gives rise to a sinusoidal transverse position $r(t) = r_\beta \sin(k_\beta ct)$

and velocity $\beta_r(t) = k_\beta r_\beta \cos(k_\beta ct)$. With $\beta^2 = \beta_r^2 + \beta_z^2$ we find:

$$
\beta_z(t) = \beta_{z0} \left( 1 - \frac{r_\beta^2 k_\beta^2}{4\beta_{z0}^2} \right) - \frac{r_\beta^2 k_\beta^2}{4\beta_{z0}^2} \cos(2k_\beta ct)
$$

and

$$
z(t) = z_0 + \beta_{z0} \left( 1 - \frac{r_\beta^2 k_\beta^2}{4\beta_{z0}^2} \right) ct - \frac{r_\beta^2 k_\beta}{8\beta_{z0}^2} \sin(2k_\beta ct)
$$

This again describes a figure-of-eight motion.

Since the restoring force in a blowout wakefield does not depend on the longitudinal position, just the radius, electrons will see the same wiggling field regardless of their phase in the wake. Also, there is no preferred oscillation plane, i.e. nothing to impose a phase relation on the wiggling electrons. This makes the realization of any temporal coherence difficult, if not impossible.

The betatron $K$ parameter is found by comparing these expressions with the respective wiggler/undulator expressions to

$$
K_\beta = \gamma c \omega_\beta r_\beta = \sqrt{\gamma / 2} c \omega_\beta r_\beta
$$
Betatron radiation With the purely electrostatic deflection, and constant $\gamma$ the radiation integral for a single electron writes as:

$$\frac{d^2W}{d\omega d\Omega} = \hbar \alpha_f \gamma_{z,0}^2 \sum_{n=1}^{\infty} \hat{\omega}^2 R_n \left[ \gamma_{z,0}^2 C_x^2 + C_z^2 \hat{\theta}^2 - 2 \gamma_{z,0} C_x C_z \hat{\theta} \cos(\phi) \right]$$

$$\gamma_{z,0} C_x = K_\beta \sum_{m=-\infty}^{\infty} J_m(\alpha_z) \left[ J_{n+2m-1}(\alpha_x) + J_{n+2m+1}(\alpha_x) \right],$$

$$C_z = 2 \sum_{m=-\infty}^{\infty} J_m(\alpha_z) J_{n+2m}(\alpha_x)$$

$$\alpha_z = \frac{n(\hat{\omega} / \hat{\omega}_n) K_\beta^2}{1 + K_\beta^2 / 2 + \hat{\theta}^2}, \quad \alpha_z = \frac{n(\hat{\omega} / \hat{\omega}_n) 2 K_\beta \cos(\phi)}{1 + K_\beta^2 / 2 + \hat{\theta}^2},$$

$$R_n = \sin^2 \left[ \pi n N_\beta \left( \hat{\omega} / \hat{\omega}_n - 1 \right) \right] / \left[ \pi n N_\beta \left( \hat{\omega} / \hat{\omega}_n - 1 \right) \right]^2$$

This is a pure wiggler spectrum, e.g. for 100 MeV electrons:
Accelerated electrons:

- have to be treated numerically ($f(\omega) = 0$ for realistic bunch parameters and x-ray emission).
- simply add up the single-electron spectra incoherently.
- trajectory of an electron injected into the wake at radius $r_\beta$ for constant acceleration field:

$$r(t) = r_0 \left( \frac{\gamma_0 \beta_0}{\gamma \beta} \right)^{1/4} \cos \left( \frac{E_0}{E_z} \left( \sqrt{2 \gamma \beta} - \sqrt{2 \gamma_0 \beta_0} \right) \right)$$

- trajectory angle:

$$\theta(t) = -\theta_0 \left( \frac{\gamma_0 \beta_0}{\gamma \beta} \right)^{1/4} \sin \left( \frac{E_0}{E_z} \left( \sqrt{2 \gamma \beta} - \sqrt{2 \gamma_0 \beta_0} + \frac{\pi}{4} \right) \right)$$

- Giving rise to the following on-axis spectrum:

$$\frac{d^2 W_{\text{acc}}}{d\omega d\Omega}_{\theta=0} = \frac{3e^2}{8\pi^3 \varepsilon_0 c} \int_{\gamma_i}^{\gamma_f} \gamma^2 \xi^2 K_{2/3} \left( \frac{\xi}{2} \right) d\gamma, \quad \xi(\gamma) = \frac{\omega}{3\gamma^2 K_\beta(\gamma) \omega_\beta(\gamma)}$$

- Of course, off-axis scalings as in undulator/wiggler/Thomson radiation still apply!
Numerical simulation

Betatron radius (μm)

Time (ps)

Electron Energy (MeV)

Photon Energy (keV)

Rad. Intensity × γ^2 (10^{-14} eV s sr^{-1})

Rad. Intensity × γ^2 (10^{-14} eV s sr^{-1})

\[ a_0 = 2 \quad n_e = 5 \times 10^{18} \text{cm}^{-3} \]

\[ r_0 = 2 \mu\text{m} \quad r_0 = 1 \mu\text{m} \quad r_0 = 0.5 \mu\text{m} \]

\[ r_0 = 1 \mu\text{m} \quad n_e = 10 \times 10^{18} \text{cm}^{-3} \]

\[ a_0 = 2 \quad a_0 = 1 \]

\[ \epsilon_c = 30 \text{keV} \]

\[ \epsilon_c = 10.4 \text{keV} \]

\[ \epsilon_c = 13.6 \text{keV} \]

\[ \epsilon_c = 6.4 \text{keV} \]

\[ \epsilon_c = 8.4 \text{keV} \]

\[ \epsilon_c = 3.9 \text{keV} \]

\[ \epsilon_c = 2.5 \text{keV} \]

\[ \epsilon_c = 0.7 \text{keV} \]
Bunch effects:

Non-uniform energy/angle distributions further smear out the spectrum.

\[
\frac{d^2 W_{acc}}{d\omega d\Omega} \bigg|_{\theta=0} = \frac{3e^2}{8\pi^3\varepsilon_0c} \sum_i N_{e,i} \int_{\gamma_{0,i}}^{\gamma_{f,i}} \gamma^2 \xi^2 K^{2/3} \left( \frac{\xi}{2} \right) d\gamma
\]

Dashed lines: radiated power for each spectrum
Electron oscillations have been observed...


...and can be controlled

by pulse-front-tilt:

by asymmetric focus:
Schnell et al., Nat. Comm. 4 2421 (2013)
Detection: X-ray CCD as imager and spectrometer (below 30-40 keV)

**Imaging mode:**

- Requires many photons/pixel for low-noise image
- Spatial resolution given by pixel size and projection factor
- Absorption and phase-contrast imaging possible
- Can be combined with filter-based spectrometers to give spectral information

**Single-hit spectroscopy**

- Needs guaranteed less than one photon/pixel (not more than 1% of pixels see signal)
- Single pixel value gives energy for single photon
- Histogram of all pixels gives spectrum
- Spatial resolution given by necessary binning for spectrum statistics
First observation at SLAC:

From LWFA beams:
Betatron radiation source characteristics

peaks at 5.5 keV

assuming a 5-fs pulse duration, this infers a peak brilliance of 
2 x 10^{22} \text{ph}/(\text{s}^2 \text{mm}^2 \text{mrad}^2 0.1\% \text{ bandwidth})

J. Wenz et al., Nat. Comm. 6 7568 (2015)
Angular resolved photon energy
Betatron applications: 1. LWFA diagnostic
Poor man’s plasma diagnostics:

Electron injection:
- How large is injection radius?
- Is injection polarization dependent?

→ Knife-edge diffraction directly yields average betatron radius $r_\beta$. With $r_0 = r_\beta \gamma^{1/4}$ * follows injection radius $r_0$.

Similar information by analysis of spectrum and spectrum model simulations retrieved by**
(not quite so poor man-wise...)

Transverse fields:
- How strong are transverse wakefields?
- What is the average wiggling parameter?

→ Beam divergence is $K/\gamma$. From measured beam divergence and measured spectrum (and some modeling) one gets effective $K$ and effective fields. Unisotropy gives information on wiggling (and injection) plane.

*S. Corde et al., Rev. Mod. Phys. 85 1 (2013)
Detailed electron trajectory modelling can match the experimental X-ray profiles

Application to a gas cell laser–plasma accelerator

A transverse displacement can only be achieved through a long capillary, which is impractical in gas cell lasers due to the high pressures involved. Indeed, under this configuration, the plasma parameters are very close to those of the laser pulse. In this case, the x-ray emission is mainly due to the electron injection position with a reduced wake velocity and wave-breaking threshold, which favors the inverse magnetic reconnection process.

As a result, variations of the emission length are the main source of the asymmetry observed on some shots in our configuration. The asymmetry is usually reduced at higher densities, because both the dephasing and the depletion lengths decrease with the electron density.

The experiment was conducted at Laboratoire d'Optique Appliquée à l'Energie Intense (LOAEI) using a high-contrast, non-collinear double-crest filamentation scheme to deliver 0.9 J of laser energy on target with a full-width at half-maximum of 1.2 cm.

The imprint of the capillary exit in the x-ray beam profile is measured. Each x-ray image has a 2 cm × 2 cm size. The size of the aperture shadow on the detector, rays coming from a longitudinal source line, and the measurement of the gradient length is dominated by the longitudinal extension of the signal radial profile in the detector plane.

The size of the aperture shadow on the detector, rays coming from a longitudinal source line, and the measurement of the gradient length is dominated by the longitudinal extension of the signal radial profile in the detector plane. In the few tens of pC range, Figure 4 shows two different emission positions in the plasma, as a result of the opening angle associated with the capillary exit, the exit pressure, and the line axis. The size of the capillary can simplify the x-ray beam profile d(z) as a function of the emission length z.
Betatron applications: 2. Imaging
The intensity distribution on the detector is a result of wavefront distortions introduced by phase object. The Transport of Intensity Equation relates sample thickness to measured intensity distribution:

\[
T(\mathbf{r}) = -\frac{1}{\mu_{\text{poly}}} \times \ln \left\{ IFT \left( \frac{\hat{I}(v/u \cdot \mathbf{r})}{l_0} \frac{l_0}{1 + \frac{(v-u)\delta_{\text{poly}}}{v/u \mu_{\text{poly}} \left| \mathbf{k} \right|^2}} \right) \right\}
\]

\(\delta_{\text{poly}}\) and \(\mu_{\text{poly}}\) are polychromatic refraction and absorption coefficients, respectively.
Results extremely important for:
Designing future accelerators
Compact X ray source (Thomson, Compton, Betatron, or FEL)
Applications (chemistry, radiotherapy, medicine, material science, ultrafast phenomena studies, etc...)

First X rays betatron contrast images

S. Fourmaux et al., Opt. Lett. 36, 13 (2011)

E. Esarey et al., Rev. Mod. Phys. 81, 1229 (2009)
S. Corde et al., Rev. of Modern Physics 85, 1 (2013)
The transport-of-intensity-equation (TIE) relates the edge-enhanced image at the detector (a) to the phase map of the insect (b).

tomographic reconstruction of 2-D projections yields cuts through sample (edge enhancement (a) and phase images (b,c))
3D rendering of the fly (with S. Schleede, F. Pfeiffer et al., TUM)

- Demonstrates suitability for high-resolution imaging (approx. 30 µm voxel or better) for an all-optical source
- Photon energies for human diagnosis require 10J-class laser; long scan times.

J. Wenz et al., Nat. Comm. 6 7568 (2015)
Tomographic 3D reconstruction of human trabecular bone

Projection at 1 degrees

Sinogram

Degrees
Tomographic 3D reconstruction of human trabecular bone

- Voxel size: $4.8 \times 4.8 \times 4.8 \, \mu m$
  - Limited by geometric magnification
  - Resolution $\approx 50 \, \mu m$
- Total scan time 4 hours
  - @ 10 Hz laser operation this image could be achieved in 3.6 seconds
- Total dose $\approx 40 \, mGy$
  - Potential for in-vivo studies
- Data quality already suitable for studies of osteoporosis

- Betatron applications: 3. ultrafast studies
Shock Imaging Setup

Wakefield driver: (12.2 +/- 0.3) J, 45 fs, 800 nm.
Shock driver: (15.7 +/- 1.0) J, 1.5 ns, 800 nm.
Max Intensity ≈ 2x10^{13} Wcm^{-2}. Material Pressure ≈ 10-20 GPa.
Target magnification = 29.
Variable betatron probe delay.
Ran FLASH simulations for using experimental drive spatial profile, but at normal incidence rather than $30^\circ$.

Simulated 2 ns top hat pulse whereas the experimental pulse was more complicated.

FLASH neglects material strength.

J. Wood, S.P.D. Mangles, Z. Najmudin, private communication
Variables:

\( \vec{r}, x, y, z \) : Position vector and its components
\( \lambda_{\alpha, \beta, L, x, y, z} \) : Wavelength of undulator, betatron, laser, x-ray oscillation
\( \omega_{\alpha, \beta, L, x, y, z} \) : frequency of undulator, betatron, laser, x-ray oscillation
\( k_{\alpha, \beta, L, x, y, z} \) : wave number of undulator, betatron, laser, x-ray oscillation
\( K_{\alpha, \beta} \) : Undulator/Betatron strength parameter
\( \gamma, \gamma_{\perp}, \gamma_{\parallel} \) : Rel. \( \gamma \)-factor, \( \perp / \parallel \) to main axis
\( \vec{\beta}, \vec{\beta}_{x,y,z,\perp,\parallel} \) : norm. velocity in \( x,y,z \) direction or \( \perp / \parallel \) to main axis
\( a_0, a(t) \) : dimensionless laser amplitude, max. or temporal evolution
\( \alpha_f \) : (Sommerfeld's) fine structure constant
\( \vec{E}_{0, x,y,z} \) : Electric field amplitude or component in \( x,y,z \) direction
\( \vec{B}_{0, x,y,z} \) : Magnetic field amplitude or component in \( x,y,z \) direction
\( P, P_R \) : Power, radiation power
\( \vec{p}, \vec{p}_{x,y,z,\perp,\parallel} \) : Momentum, in \( x,y,z, \perp, \parallel \) orientation
\( P_\mu \) : 4-momentum
\( t, \tau \) : time, proper time
\( \dot{\vec{\beta}} \) : time derivative of norm. velocity → norm. acceleration
\( \vec{v} \) : non-normalized velocity
\( E \) : non-vector: Energy
\( \Omega \) : solid angle
\( \theta \) : angle from observation direction
\( \phi \) : azimuthal angle
\( \varphi \) : phase
\( \tilde{E}, \tilde{B} \) : tilde denotes frequency domain
\( q, e \) : charge, elementary charge
\( R \) : observation distance
\( \vec{n} \) : unit vector pointing towards observer
\( \xi \) : normalized frequency in observation direction
\( \omega_c \) : critical frequency
\( K_n, J_n \) : Bessel functions
\( \omega' \) : Frequency in inner frame of electron
\( R \left( \frac{N \Delta \omega}{\omega_1} \right), R_n \) : Shape function of fundamental harmonic
\( N_{\alpha, \beta} \) : Number of oscillations in undulator/laser/betatron field
\( N_e \) : Number of electrons in bunch
\( r_\beta \) : max. oscillation amplitude, betatron radius
\( \hat{\omega}, \hat{\omega}_n \) : normalized frequency: \( \hat{\omega} = \omega / 2 \gamma z_0 \omega_{L/\beta}, \hat{\omega}_n = \omega / 2 \gamma^2 z_0 \omega_{L/\beta} \)
\( \hat{\theta} \) : normalized angle: \( \hat{\theta} = \gamma z_0 \theta \)
\( \hat{\vartheta} \) : collision angle
Thank you!
Tomography: Line projections and Radon transform:

Parametrize each point on ray by a normal unit vector \( \omega \), distance to rotation center \( t \) and longitudinal position \( s \):

\[
f(x) = f\left(\omega t + s \omega^\perp\right)
\]

Then the Radon transform yields a representation of the object function \( f \) in the variables \( t \) and \( \omega \):

\[
Rf(t, \omega) = \int_{x \cdot \omega = t} f(x) \, dx = \int_{-\infty}^{\infty} f\left(\omega t + s \omega^\perp\right) \, ds
\]

\( \omega \) normal unit vector, \( t \) rotation center, \( s \) longitudinal position, \( x \) point on ray, \( \omega^\perp \) orthogonal vector.
Tomography:

- Projections are \((n-1)\)-dim. distribution functions representing the line integrals of the \(n\)-dim. density distribution along each ray path.
- The set of projections under different angles \(\alpha\) constitute a sinogram:

\[
f(\bar{x})
\]

\[
Rf(t, \omega_n)
\]
Reconstruction: Inversion of Radon transform:

Overlapping backprojections

Filtered backprojection formula:

\[ f = \frac{1}{4\pi} R^\# \left( \frac{d}{dt} \langle Rf \rangle \right) \]

Filtered

\[ H(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{y-x} \, dx \]
Is it so simple? Can one use freely available undulator codes to predict a Compton X-ray source?

Remember: Undulator is purely magnetic, light is electro-magnetic

Numerical check (Undulator model vs. exact Thomson formula):

![Graphs showing frequency mismatch and intensity error for fundamental, third, and fifth harmonics as functions of electron energy and field strength.](image)